JOURNAL OF FUNCTIONAL ANALYSIS 30, 358–389 (1978)

Spectral Analysis of Families of Operator Polynomials and a Generalized Vandermonde Matrix II: The Infinite Dimensional Case

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Communicated by the Editors

Received July 22, 1977

This paper deals with the problem of the existence of a common monic multiple for a given family of monic operator polynomials. In general such a multiple does not exist. Necessary and sufficient conditions for the existence are given in terms of the invertibility of a generalized Vandermonde operator matrix. The minimal possible degree of the multiple is determined and the role played by the spectral properties of the divisors is explained. The problems considered here have their origin in the spectral theory of non-self-adjoint operators.

INTRODUCTION

Let $L_1, \ldots, L_r$ be monic polynomials whose coefficients are bounded linear operators acting on a Banach space. In this paper we consider the problem of the existence of a monic operator polynomial L such that $L_1, \ldots, L_r$ are right divisors of L, and in case such a multiple L exists, the relation between L and its divisors is studied. In particular, we determine the minimal possible degree of the multiple and we describe the role played by the special properties of the divisors. The study of problems of this type has its origin in the spectral

0022-1236/78/0303–0358$02.00/0

theory of non-self-adjoint operators (e.g., see [5, Chap. V]). Especially, we would like to refer to Krein and Langer's paper [10], which deals with the mathematical principles of small vibrations of a continuum. Markus and Mereutsa [11] have developed a method to deal with the above-mentioned problems for the case of polynomials of degree 1. Some generalizations of their results have been announced in [9].

A complete study of the finite-dimensional case has been carried out by the authors in [4]. In the present paper, the infinite-dimensional case is considered. Here many new difficulties arise and the final results differ considerably from those of [4]. As in [4], the main tool is a generalized Vandermonde operator matrix, but in this paper, we do not require the Vandermonde to be a square matrix. Further, we show that not only is the invertibility of the Vandermonde important, but also some form of generalized invertibility.

To illustrate our results, we formulate here a corollary of the main theorem for polynomials of degree one. Let $L_j(\lambda) = \lambda I - T_j$ ($j = 1,\dots, r$), where $T_1, \dots, T_r$ are bounded linear operators on a Banach space $\mathfrak{B}$, and for $m = 1, 2,\dots$ put

$$V_m = \begin{bmatrix} I \cdots I \\ T_1 \cdots T_r \\ \cdots\cdots \\ T_1^{m-1} \cdots T_r^{m-1} \end{bmatrix}$$

Then Ker $V_m =$ Ker V_{m+1} is a necessary condition for the existence of a common monic left multiple L of degree m for $L_1, \dots, L_r$. This result allows us to construct simple examples of polynomials which do not have a common left multiple. More complicated examples are given to show that under the condition Ker $V_m =$ Ker V_{m+1} it could happen that $L_1, \dots, L_r$ have no common multiple of degree m with bounded coefficients, but that a multiple of degree m with unbounded coefficients exists. On the other hand, if Im V_m is closed, and Ker V_m and Im V_m have both a closed complement, then the condition Ker $V_m =$ Ker V_{m+1} is also sufficient for the existence of a common multiple of degree m. In that case, the multiple may be constructed as follows. Take a generalized inverse W of V_m (i.e., $WV_mW = W$ and $V_mWV_m = V_m$), write $W = (W_1 \cdots W_m)$ where $W_j \colon \mathfrak{B} \to \mathfrak{B}^m$ for $j = 1,\dots, m$ and put $T_m = (T_1{}^m \cdots T_r{}^m)$. Then

$$L(\lambda) = \lambda^m I - T_m(W_1 + W_2\lambda + \cdots + W_m\lambda^{m-1})$$

is a left multiple of $L_1, \dots, L_r$. Moreover, if V_m is right invertible, then this polynomial is the only left multiple of $L_1, \dots, L_r$ of degree m.

This paper consists of 11 sections. In Section 2 the Vandermonde operator is introduced. Using the theory of monic operator polynomials, as developed in [8], we show that the Vandermonde operator appears in a natural way in certain interpolation problems and in problems concerning the existence of multiples. In Section 2 also appears the first example showing that a common multiple does not always exist. The third section deals with questions about the uniqueness of the multiple. In Section 4 the existence of a common multiple is proved for a system of monic operator polynomials whose Vandermonde is a Fredholm operator.

The main theorem about the existence of a common multiple and the description of the minimal possible degree are contained in Section 5. An example is given to illustrate the main theorem, and for the special case that the polynomials have disjoint spectra the conditions of the main theorem are shown to be necessary too. An important example about the absence of a common multiple is given in Section 6. In the seventh section we present an alternative, more geometric construction of the multiple in terms of standard pairs.

In Sections 8–11 we assume the existence of a common multiple and we study the relationships between the properties of the original polynomials and those of the multiple. In particular, we study the effect on the multiple of certain spectral conditions on the divisors. Among other things, this allows us to see that the conditions of the main theorem about the existence of a common multiple cannot be weakened much further.

1. Preliminaries

In this paper, the word operator is used for a bounded linear operator acting between Banach spaces, unless the contrary is mentioned explicitly. Given a Banach space $\mathfrak{A}$, the symbol $\mathfrak{A}^{\ell}$ will denote the direct sum of ℓ copies of $\mathfrak{A}$ endowed with the usual, normable topology. Operators from $\mathfrak{A}^{\ell_1}$ into $\mathfrak{A}^{\ell_2}$ will often be written as $\ell_2 \times \ell_1$-matrices whose entries are operators acting on $\mathfrak{A}$. As in [4] we shall use the notations $\operatorname{row}(B_j)_{j=1}^{\ell}$, $\operatorname{col}(B_j)_{j=1}^{\ell}$ and $\operatorname{diag}(B_j)_{j=1}^{\ell}$ to denote the operators $[B_1 \cdots B_\ell]\colon \mathfrak{A}^{\ell} \to \mathfrak{A}$,

$$\begin{bmatrix} B_1 \\ \cdots \\ B_\ell \end{bmatrix} : \mathfrak{A} \to \mathfrak{A}^{\ell}, \qquad \begin{bmatrix} B_1 & 0 & & 0 \\ 0 & B_2 & \cdots & 0 \\ 0 & 0 & & B_\ell \end{bmatrix} : \mathfrak{A}^{\ell} \to \mathfrak{A}^{\ell},$$

respectively.

Throughout this paper $\mathfrak{B}$ will be a complex Banach space, and except when the contrary is stated explicitly, we shall suppose that the coefficients of each operator polynomial considered in this paper are operators on $\mathfrak{B}$. An operator polynomial $L(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^{\ell} A_\ell$ is said to have *degree* ℓ if $A_\ell \neq 0$, and L is called *monic* if A_ℓ is the identity operator I on $\mathfrak{B}$. In this paper, we shall use certain elements of the theory of monic operator polynomials as developed in [8], which we shall recall here briefly.

Two operators $X: \mathfrak{B}^\ell \to \mathfrak{B}$, $T: \mathfrak{B}^\ell \to \mathfrak{B}^\ell$ are said to form a *standard pair* (X, T) of *degree* ℓ if $\operatorname{col}(XT^{i-1})_{i=1}^\ell$ is a (two-sided) invertible operator on $\mathfrak{B}^\ell$.

We call the standard pair (X, T) a standard pair of the monic operator polynomial $L(\lambda) = \lambda^\ell I + \lambda^{\ell-1}A_{\ell-1} + \cdots + A_0$ if its degree is ℓ and

$$XT^\ell + A_{\ell-1}XT^{\ell-1} + \cdots + A_0X = 0.$$

In that case L admits the following representation:

$$L(\lambda) = \lambda^\ell I - XT^\ell(U_1 + U_2\lambda + \cdots + U_\ell\lambda^{\ell-1}), \tag{1.1}$$

where $\operatorname{row}(U_j)_{j=1}^\ell = [\operatorname{col}(XT^{i-1})_{i=1}^\ell]^{-1}$. Given L, one can always find a standard pair of L. In fact, one may choose $([I\ 0\ \cdots\ 0], C_L)$, where C_L is the (first) companion operator of L, i.e.,

$$C_L = \begin{bmatrix} 0 & I & & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \cdots & -A_{\ell-1} \end{bmatrix}.$$

The representation (1.1) is called the (right) *normal* form of L. Any two standard pairs (X_1, T_1) and (X_2, T_2) of L are *similar*, i.e., $X_1 = X_2S$, $T_1 = S^{-1}T_2S$ for some invertible operator S on $\mathfrak{B}^\ell$. In the sequel we shall often use the following division theorem (see [8]).

THEOREM 1.1. *Let $L(\lambda) = \lambda^m I + \lambda^{m-1}A_{m-1} + \cdots + A_0$ be a monic operator polynomial, and let $L_1(\lambda) = \lambda^k I - X_1T_1{}^k(U_{11} + \lambda U_{12} + \cdots + \lambda^{k-1}U_{1k})$ be a monic operator polynomial in normal form. Suppose $k \leqslant m$. Then the remainder of L after division on the right by L_1 is equal to*

$$\sum_{j=1}^k \lambda^{j-1}(A_0X_1U_{1j} + \cdots + A_{m-1}X_1T_1^{m-1}U_{1j} + X_1T_1^m\, U_{1j}).$$

Note that a linear polynomial $L_1(\lambda) = \lambda I - B$ is a right divisor of $L(\lambda) = \lambda^m I + \lambda^{m-1}A_{m-1} + \cdots + A_0$ if and only if B is a right operator *root*, i.e.,

$$A_0 + A_1B + \cdots + A_{m-1}B^{m-1} + B^m = 0.$$

In this paper we shall use the following form of generalized invertibility. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Banach spaces, and let $A: \mathfrak{A}_1 \to \mathfrak{A}_2$ be an operator. An operator $A^+: \mathfrak{A}_2 \to \mathfrak{A}_1$ will be called a *generalized inverse* for A if

$$AA^+A = A, \qquad A^+AA^+ = A^+.$$

If A^+ is such an operator, then A^+A and AA^+ are (bounded linear) projectors, $\operatorname{Ker} A^+A = \operatorname{Ker} A$, $\operatorname{Im} AA^+ = \operatorname{Im} A$, and hence $\operatorname{Ker} A$ and $\operatorname{Im} A$ are complemented subspaces of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively. Conversely, if $\operatorname{Ker} A$ and $\operatorname{Im} A$ are complemented subspaces of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, then A has a generalized inverse. In particular, if A acts between finite-dimensional spaces, then a generalized inverse A^+ always exists. The following lemma will be useful later.

LEMMA 1.2. *Let $\mathfrak{A}$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$ be Banach spaces, and let $A_1: \mathfrak{A} \to \mathfrak{B}_1$ and $A_2: \mathfrak{A} \to \mathfrak{B}_2$ be operators. Put $A = \operatorname{col}(A_i)_{i=1}^2$.*

(1) *If A_1 has a generalized inverse $A_1{}^+$ and* $\operatorname{Ker} A_1 \subset \operatorname{Ker} A_2$, *then $(A_1{}^+ \ 0)$ is a generalized inverse for A.*

(2) *If A has a generalized inverse $A^+ = (B_1 B_2)$ and $XA_1 = A_2$ for some operator $X: \mathfrak{B}_1 \to \mathfrak{B}_2$, then $B_1 + B_2 X$ is a generalized inverse for A_1.*

Proof. Make the necessary computations.

An operator $A: \mathfrak{A}_1 \to \mathfrak{A}_2$ is called *left invertible* if VA is the identity operator on $\mathfrak{A}_1$ for some operator $V: \mathfrak{A}_2 \to \mathfrak{A}_1$. If for some $V: \mathfrak{A}_2 \to \mathfrak{A}_1$ the operator AV is the identity operator on $\mathfrak{A}_2$, then A is said to be *right invertible*. We call A *regular* if for some $\alpha > 0$ we have $\| Ax \| \geqslant \alpha \| x \|$ for all $x \in \mathfrak{A}_1$. Left invertibility of A means the same as $\operatorname{Ker} A = (0)$ and $\operatorname{Im} A$ is (closed and) complemented, while the regularity of A is equivalent to $\operatorname{Ker} A = (0)$ and $\operatorname{Im} A$ is closed. If $\mathfrak{A}_2$ is a Hilbert space, then the two concepts are the same. Note that right invertibility of A is equivalent to $\operatorname{Im} A = \mathfrak{A}_2$ and $\operatorname{Ker} A$ is complemented.

For an operator polynomial $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ the *spectrum* $\sigma(L)$ of L is defined to be the set of all complex numbers λ such that $L(\lambda)$ is not invertible. If the leading coefficient A_ℓ is invertible, then $\sigma(L)$ is a compact subset of $\mathbb{C}$. If $L(\lambda) = \lambda I - A$, then $\sigma(L)$ is equal to the spectrum of the operator A.

2. THE VANDERMONDE OPERATOR AND EXISTENCE OF COMMON MULTIPLES

Let $L_1, \ldots, L_r$ be monic operator polynomials of degrees $k_1, \ldots, k_r$, respectively, and for $1 \leqslant j \leqslant r$ let (X_j, T_j) be a standard pair of L_j. We want to answer the following question: do $L_1, \ldots, L_r$ have a common monic left multiple? In other words, does there exist a monic operator polynomial L such that $L_1, \ldots, L_r$ are right divisors of L?

Let $L(\lambda) = \lambda^m I + \lambda^{m-1} A_{m-1} + \cdots + A_0$. Then in view of Theorem 1.1 the polynomial L will be a common left multiple of $L_1, \ldots, L_r$ if and only if the following identity holds true.

$$[A_0\ A_1\ \cdots\ A_{m-1}]\begin{bmatrix} X_1U_1 & X_2U_2 & \cdots & X_rU_r \\ X_1T_1U_1 & X_2T_2U_2 & \cdots & X_rT_rU_r \\ \cdots & \cdots & \cdots & \cdots \\ X_1T_1^{m-1}U_1 & X_2T_2^{m-1}U_2 & \cdots & X_rT_r^{m-1}U_r \end{bmatrix}$$

$$= -[X_1T_1{}^mU_1\ X_2T_2{}^mU_2\ \cdots\ X_rT_r{}^mU_r]. \tag{2.1}$$

Here $U_j = [\operatorname{col}(X_jT_j^{i-1})_{i=1}^{k_j}]^{-1}$ for $j = 1, 2, \ldots, r$. So $L_1, \ldots, L_r$ have a common monic multiple of degree m if and only if Eq. (2.1) has a solution $[A_0A_1 \cdots A_{m-1}]$. The operator $(X_jT_j^{i-1}U_j)_{i=1,j=1}^{m\ \ r}$, which appears as the second factor in the left-hand side of (2.1) is called the (generalized) *Vandermonde operator* of *order* m of the polynomials $L_1, \ldots, L_r$ and will be denoted by $V_m(L_1, \ldots, L_r)$.

If $L_j(\lambda) = \lambda I - Z_j$ for $1 \leqslant j \leqslant r$, then $V_m(L_1, \ldots, L_r)$ is equal to the following operator:

$$\begin{bmatrix} I & I & \cdots & I \\ Z_1 & Z_2 & \cdots & Z_r \\ \cdots & \cdots & \cdots & \cdots \\ Z_1^{m-1} & Z_2^{m-1} & \cdots & Z_r^{m-1} \end{bmatrix}.$$

This operator matrix is the Vandermonde operator as introduced by Markus and Mereutsa in [11]. For the case that $m = k_1 + \cdots + k_r$ and $\dim \mathfrak{B} < \infty$, the Vandermonde operator has been introduced in [4]. The infinite-dimensional case is also considered in [9].

The definition of $V_m(L_1, \ldots, L_r)$ does not depend on the special choice of the standard pair $(X_1, T_1), \ldots, (X_r, T_r)$. In fact, the Vandermonde can be expressed explicitly in terms of the coefficients of $L_1, \ldots, L_r$ (see [4, Theorem 2.2] and note that this theorem also holds in the infinite-dimensional case).

The Vandermonde operator also appears in a natural way in certain interpolation problems. Let $L_1, \ldots, L_r$ be as before, and suppose that operator polynomials $R_1, \ldots, R_r$ of the form

$$R_j(\lambda) = R_{j0} + \lambda R_{j1} + \cdots - \lambda^{k_j-1}R_{j,k_j-1}$$

are given. Now one may ask whether there exists a monic operator polynomial L such that L after division on the right by L_j yields R_j as a remainder ($1 \leqslant j \leqslant r$). If $R_j = 0$ for $j = 1, \ldots, r$, then such a polynomial L will be a common multiple of $L_1, \ldots, L_r$. If $k_1 = \cdots = k_r = 1$, then the question is related to the interpolation problem discussed in [3]. Again using Theorem 1.1, one sees that $L(\lambda) = \lambda^mI + \lambda^{m-1}A_{m-1} + \cdots + A_0$ is a solution of the above problem if and only if

$$[A_0 \cdots A_{m-1}]\ V_m(L_1, \ldots, L_r) = \operatorname{row}(-X_jT_j{}^mU_j + S_j)_{j=1}^r,$$

where $S_j = [R_{j0} \cdots R_{jk_j-1}]$. In the sequel we shall deal with Eq. (2.1) only, but our methods can be used to solve the above equation too.

In order that (2.1) is solvable, it is necessary that

$$\operatorname{Ker} V_m(L_1, \ldots, L_r) \subset \operatorname{Ker}[\operatorname{row}(X_j T_j^m U_j)_{j=1}^r]. \tag{2.2}$$

Let us write V_m for $V_m(L_1, \ldots, L_r)$ whenever there is no danger of misunderstanding. As $\operatorname{row}(X_j T_j^m U_j)_{j=1}^r$ is the last operator row in V_{m+1}, one sees that (2.2) is equivalent to the requirement that $\operatorname{Ker} V_m = \operatorname{Ker} V_{m+1}$. Now the last identity implies that $\operatorname{Ker} V_m = \operatorname{Ker} V_{m+p}$ for $p = 1, 2, \ldots$. This leads to the following definition:

$$\operatorname{ind}_1(L_1, \ldots, L_r) = \inf\{m > 0 \mid \operatorname{Ker} V_m = \operatorname{Ker} V_{m+p} \text{ for } p \geqslant 1\}.$$

Here, as usual, $\inf \varnothing = \infty$. We call this number the (first) *index of stabilization* of the family $L_1, \ldots, L_r$ (cf. [4, Section 13]). From the preceding discussion, we conclude that $\operatorname{ind}_1(L_1, \ldots, L_r) < \infty$ is a necessary condition for the solvability of Eq. (2.1). It is now clear how to construct two monic operator polynomials which do not have a common monic left multiple.

EXAMPLE 2.1. For $k = 1, 2, \ldots$ let $\mathbb{C}_k$ denote the complex Euclidean space of dimension k, and define operators A_k and B_k on $\mathbb{C}_k$ by setting

$$A_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad B_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Let H be the Hilbert space direct sum $\mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{C}_3 \oplus \cdots$, and put $A = A_1 \oplus A_2 \oplus A_3 \oplus \cdots$, $B = B_1 \oplus B_2 \oplus B_3 \oplus \cdots$. Then A and B are operators on H and

$$V_m = V_m(\lambda I - A, \lambda I - B) = \operatorname{col}(A^i B^i)_{i=0}^{m-1}.$$

So $\operatorname{Ker} V_m$ consists of all pairs $((x_1, x_2, \ldots), (-x_1, -x_2, \ldots))$ in $H \times H$ such that for $k = 1, 2, \ldots$ the last $m - 1$ coordinates of x_k (as element of $\mathbb{C}_k$) are zero. It follows that $\operatorname{Ker} V_m \neq \operatorname{Ker} V_{m+1}$ for all $m \geqslant 1$. Therefore, $\operatorname{ind}_1(\lambda I - A, \lambda I - B) = \infty$, and A and B cannot be right roots of any monic operator polynomial. Or in other words, the monic operator polynomials $\lambda I - A$ and $\lambda I - B$ do not have a common monic left multiple.

THEOREM 2.2. *Let $L_1, \ldots, L_r$ be monic operator polynomials. A necessary condition for the existence of a monic common left multiple of $L_1, \ldots, L_r$ of degree m is that* $\operatorname{ind}_1(L_1, \ldots, L_r) \leqslant m$. *This condition is also sufficient if $V_m(L_1, \ldots, L_r)$ has a generalized inverse.*

Proof. The necessity of $\operatorname{ind}_1(L_1, ..., L_r) < \infty$ has been shown above.

Suppose now that $m \geqslant \operatorname{ind}_1(L_1, ..., L_r)$, and let W be a generalized inverse of $V_m = V_m(L_1, ..., L_r)$. As $m \geqslant \operatorname{ind}_1(L_1, ..., L_r)$, we know that formula (2.2) holds true. Further, the fact that W is a generalized inverse of V_m implies that $\operatorname{Im}(I - WV_m) = \operatorname{Ker} V_m$. So we have

$$[\operatorname{row}(X_j T_j^m U_j)_{j=1}^r] W V_m = \operatorname{row}(X_j T_j^m U_j)_{j=1}^r . \tag{2.3}$$

Note that $[\operatorname{row}(X_j T_j^m U_j)_{j=1}^r] W$ is an operator from $\mathfrak{B}^m$ into $\mathfrak{B}$. Write this operator as $[A_0 A_1 \cdots A_{m-1}]$. Then we see from (2.3) that

$$[A_0 A_1 \cdots A_{m-1}]\, V_m = \operatorname{row}(X_j T_j^m U_j)_{j=1}^r ,$$

and hence $L(\lambda) = \lambda^m I + \lambda^{m-1} A_{m-1} + \cdots + A_0$ is a common left multiple of $L_1, ..., L_r$.

The condition in the second part of Theorem 2.2 that $V_m(L_1, ..., L_r)$ has a generalized inverse may be replaced by the weaker condition that $V_m(L_1, ..., L_r)$ has complemented range. Indeed, in that case the operators $A_0, A_1, ..., A_{m-1}$ may be defined in the following way. Take x in $\operatorname{Im} V_m(L_1, ..., L_r)$, and put

$$[A_0 A_1 \cdots A_{m-1}]\, x = [\operatorname{row}(X_j T_j^m U_j)_{j=1}^r]\, y,$$

where y is chosen such that $V_m(L_1, ..., L_r)\, y = x$. Then, as $m \geqslant \operatorname{ind}_1(L_1, ..., L_r)$, formula (2.2) guarantees that $[A_0 A_1 \cdots A_{m-1}]$ is well defined on $\operatorname{Im} V_m(L_1, ..., L_r)$. Next, we define $[A_0 A_1 \cdots A_{m-1}]$ to be zero on some direct complement of $\operatorname{Im} V_m(L_1, ..., L_r)$ in $\mathfrak{B}^m$. The operators $A_0, A_1, ..., A_{m-1}$ defined in this way satisfy Eq. (2.1).

The next example shows that the condition that $V_m(L_1, ..., L_r)$ has a generalized inverse is not necessary to guarantee the existence of a common multiple of degree m.

EXAMPLE 2.3. Let B be a compact operator acting on the infinite-dimensional Banach space $\mathfrak{A}$, and suppose that $\operatorname{Ker} B = (0)$. Put $L_1(\lambda) = \lambda I$ and $L_2(\lambda) = \lambda I - B$, where I is the identity operator on $\mathfrak{A}$. As

$$\operatorname{Ker} \begin{bmatrix} I & I \\ 0 & B \end{bmatrix} = (0),$$

we have $\operatorname{ind}_1(L_1, L_2) = 2$, and since $\operatorname{Im} \operatorname{col}(B^j)_{j=1}^{m-1}$ is not closed for any m, the Vandermonde operator $V_m(L_1, L_2)$ has no generalized inverse for any m. On the other hand, $L_1(\lambda)$ and $L_2(\lambda)$ commute, and thus $\lambda^2 I - \lambda B$ is a common left multiple of L_1 and L_2. Note that the degree of the multiple is precisely $\operatorname{ind}_1(L_1, L_2)$.

COROLLARY 2.4. *If $V_m(L_1, \ldots, L_r)$ is left invertible, then there exists a monic common multiple of $L_1, \ldots, L_r$ of degree m.*

Proof. Apply Theorem 2.2.

For linear divisors and $m = k_1 + \cdots + k_r$ (where k_j is the degree of L_j) Corollary 2.4 has been proved by Markus and Mereutsa in [11].

As in [4], one may introduce the notion of a left Vandermonde operator for the monic operator polynomials $L_1, \ldots, L_r$:

$$V_m^{\text{left}}(L_1, \ldots, L_r) = (Z_i T_i^{j-1} R_i)_{i=1,j=1}^{r \quad m},$$

where $Z_i = \operatorname{col}(Z_{ij})_{j=1}^{k_i}$ and

$$L_i(\lambda) = \lambda^{k_i} I - (Z_{i1} + \lambda Z_{i2} + \cdots + \lambda^{k_i - 1} Z_{ik_i})\, T_i R_i$$

(see [8, Theorem 3]; this result holds in infinite-dimensional case as well). This definition is slightly different from the one given in [4]. Note that

$$[V_m^{\text{left}}(L_1, \ldots, L_r)]^* = V_m(L_1^*, \ldots, L_r^*).$$

So $V_m^{\text{left}}(L_1, \ldots, L_r)$ is related to left divisors and right multiples in the same way as $V_m(L_1, \ldots, L_r)$ is related to right divisors and left multiples.

3. UNIQUENESS OF A MONIC COMMON MULTIPLE

Let $L_1, \ldots, L_r$ be monic operator polynomials. From Theorem 2.2 it follows that two-sided invertibility of $V_m(L_1, \ldots, L_r)$ implies the existence of a unique common monic left multiple of $L_1, \ldots, L_r$ of degree m. Although left invertibility of $V_m(L_1, \ldots, L_r)$ implies the existence of a common monic multiple of degree m (cf. Corollary 2.4), it does not necessarily imply the uniqueness. This is shown by the following example.

EXAMPLE 3.1. Take $\mathfrak{B}$ to be ℓ_2, and let $L_1(\lambda) = \lambda I$ and $L_2(\lambda) = \lambda I - S$, where S is the shift operator $S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots)$. Then

$$V_2(L_1, L_2) = \begin{bmatrix} I & I \\ 0 & S \end{bmatrix}$$

is left invertible. In this case, for $m = 2$ Eq. (2.1) takes the following form:

$$[A_0 \; A_1] \begin{pmatrix} I & I \\ 0 & S \end{pmatrix} = [0 \;\; S^2].$$

The operators A_0, A_1 are solutions of this equation if and only if $A_0 = 0$ and $A_1 |_{\operatorname{Im} S} = S |_{\operatorname{Im} S}$. Therefore, the general form of a common monic multiple L of L_1 and L_2 of degree 2 is $L(\lambda) = \lambda^2 I - \lambda A_1$, where A_1 is any operator such that $A_1|_{\operatorname{Im} S} = S|_{\operatorname{Im} S}$. In particular, we see that L_1 and L_2 have many different common monic left multiples of degree 2.

To assure the uniqueness of a common monic multiple of degree m (assuming it exists), one does not need the two-sided invertibility of $V_m(L_1,..., L_r)$. In fact, to get uniqueness it is sufficient that $V_m(L_1,..., L_r)$ is right invertible, or, somewhat weaker, that Ker $V_m(L_1,..., L_r)^* = (0)$. For monic operator polynomials of degree 1, this has already been noted by Markus and Mereutsa in [11] (cf. [9]).

To prove the above remark, suppose that Ker $V_m(L_1,..., L_r)^* = (0)$, and let $\lambda^m I + \lambda^{m-1}A_{m-1} + \cdots + A_0$ and $\lambda^m I + \lambda^{m-1}A_{m-1} + \cdots + A_0^1$ be common left multiples of $L_1,..., L_r$. Then we see from (2.1) that

$$[A_0 - A_0 \; A_1 - A_1^1 \cdots A_{m-1} - A_{m-1}]\, V_m(L_1,..., L_r) = 0.$$

From our hypothesis it follows that Im $V_m(L_1,..., L_r)$ is dense in $\mathfrak{B}^m$. So $A_j - A_j^1 = 0$ for $j = 0,..., m-1$, and the uniqueness of the multiple is proved.

Right invertibility of $V_m(L_1,..., L_r)$ does not guarantee the existence of a common monic left multiple of $L_1,..., L_r$ of degree m. In fact, $V_m(L_1,..., L_r)$ may be right invertible, while $\operatorname{ind}_1(L_1,..., L_r) > m$. This will be shown in the following example.

EXAMPLE 3.2. Take $\mathfrak{B}$ to be ℓ_2. Put $L_1(\lambda) = \lambda I - X$ and $L_2(\lambda) = \lambda I - Y$, where the operators X and Y are given by the following infinite matrices:

$$X = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & 0 & 1 & \\ & & & & 1 & 0 & \\ & & & & & & \ddots \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & 0 & & & & & \\ 1 & 0 & -1 & 0 & 0 & & & \\ & & 0 & 0 & 0 & & & \\ & & 1 & 0 & -1 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 1 & 0 & -1 & \\ & & & & & & & \ddots \end{bmatrix}.$$

Note that $X^2 = I$, $Y^2 = 0$, and $X - Y$ is a right invertible shift operator. In particular, $X - Y$ is not invertible. As

$$V_2(L_1, L_2) = \begin{bmatrix} I & I \\ X & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X - Y \end{bmatrix} \begin{bmatrix} I & I \\ I & 0 \end{bmatrix},$$

it follows that $V_2(L_1, L_2)$ is right invertible and Ker $V_2(L_1, L_2) \neq (0)$. Note that

$$V_3(L_1, L_2) = \begin{bmatrix} I & I \\ X & Y \\ I & 0 \end{bmatrix}.$$

So Ker $V_3(L_1, L_2) = (0)$. Hence, $\mathrm{ind}_1(L_1, L_2) = 3 > 2$, and by Theorem 2.2, the polynomials L_1 and L_2 do not have a common monic left multiple of degree 2.

The polynomials L_1 and L_2 introduced in the previous example do have a common monic right multiple of degree 2. In fact, as

$$V_2(L_1^*, L_2^*) = \begin{bmatrix} I & I \\ X^* & Y^* \end{bmatrix} = \begin{bmatrix} I & X \\ I & Y \end{bmatrix}^*,$$

it is easily seen that $V_2(L_1^*, L_2^*)$ is left invertible. (Here the symbol * denotes the Hilbert space adjoint.) So by Corollary 2.4, the polynomials L_1^* and L_2^* have a common monic left multiple of degree 2, and hence L_1 and L_2 have a common right multiple of degree 2.

4. Fredholm Vandermonde Operators

In this section we continue to study the problem about the existence of a common monic multiple for monic operator polynomials $L_1, \ldots, L_r$.

Theorem 4.1. *If $V_m(L_1, \ldots, L_r)$ is left invertible modulo the compact operators (i.e., for some operator Z and some compact operator S we have $ZV_m(L_1, \ldots, L_r) = I + S$), then* $\mathrm{ind}_1(L_1, \ldots, L_r)$ *is finite and $L_1, \ldots, L_r$ have a common monic left multiple of degree* $\max(m, \mathrm{ind}_1(L_1, \ldots, L_r))$.

Proof. From the conditions of the theorem, it follows that $\dim \operatorname{Ker} V_m(L_1, \ldots, L_r)$ is finite. Hence $\mathrm{ind}_1(L_1, \ldots, L_r) < \infty$. Further, as $V_m(L_1, \ldots, L_r)$ is left invertible modulo the compacts, the same is true for $V_p(L_1, \ldots, L_r)$ for every $p > m$. In particular, $V_p(L_1, \ldots, L_r)$ has a generalized inverse for each $p \geqslant m$. So we may apply Theorem 2.2 to get the desired result.

An operator A is called a *Fredholm operator* if $\dim \operatorname{Ker} A$ and $\operatorname{codim} \operatorname{Im} A$ are both finite. A point $\lambda_0 \in \mathbb{C}$ is said to be an *essential point* of the spectrum of the operator polynomial L if $L(\lambda_0)$ is not a Fredholm operator. The set of all essential points of the spectrum of L will be denoted by $\sigma_e(L)$.

Corollary 4.2. *Let $L_1, \ldots, L_r$ be monic operator polynomials of the form*

$$L_i(\lambda) = \lambda^{k_i} I + \sum_{j=0}^{k_i - 1} \lambda^j(\alpha_{ij} I + S_{ij}), \qquad i = 1, \ldots, r,$$

where α_{ij} are complex scalars and S_{ij} are compact operators. Suppose that $\sigma_e(L_i) \cap \sigma_e(L_j) = \varnothing$ for $i \neq j$. Then $\mathrm{ind}_1(L_1, \ldots, L_r)$ *is finite and there exists a common monic left multiple of $L_1, \ldots, L_r$ of degree* $\max(\sum_{j=1}^{r} k_j, \mathrm{ind}_1(L_1, \ldots, L_r))$.

Proof. Without loss of generality, we may suppose that $\mathfrak{B}$ is infinite dimensional. Put $f_i(\lambda) = \lambda^{k_i} + \sum_{j=1}^{k_i - 1} \alpha_{ij} \lambda^j$, $i = 1, \ldots, r$. As the space $\mathfrak{B}$ is infinite dimensional, $\sigma_e(L_i)$ consists of the roots of $f_i(\lambda)$, and hence the condition $\sigma_e(L_i) \cap \sigma_e(L_j) = \varnothing$ for $i \neq j$ implies (see [4, Corollary 4.4]) that $\det V_\ell(f_1, ..., f_r) \neq 0$. Here $\ell = \sum_{j=1}^{r} k_j$. Let $\tilde{L}_i(\lambda) = f_i(\lambda)I$, $i = 1, ..., r$. Then $V_\ell(\tilde{L}_1, ..., \tilde{L}_r)$ is invertible too. From Theorem 2.2 of [4] (which also holds in the infinite-dimensional case) it follows that

$$V_\ell(L_1, ..., L_r) = V_\ell(\tilde{L}_1, ..., \tilde{L}_r) + S$$

for some compact operator S. So $V_\ell(L_1, ..., L_r)$ is a Fredholm operator. In particular, $V_\ell(L_1, ..., L_r)$ is left invertible modulo the compact operators, and hence we may apply Theorem 4.1 to complete the proof.

Suppose that $\mathfrak{B} = \mathfrak{A}^m$ for some Banach space $\mathfrak{A}$ and assume that the coefficients of $L_1, ..., L_r$ are of the form $(Z \otimes I_{\mathfrak{A}}) + S$, where Z is an $n \times n$ matrix with scalar entries and S is a compact operator on $\mathfrak{B}$. Then, as in Corollary 4.2, the condition $\sigma_e(L_i) \cap \sigma_e(L_j) = \varnothing$ for $i \neq j$ implies that $L_1, ..., L_r$ have a common monic left multiple. To see this, one can use the same arguments as in the proof of Corollary 4.2, except in place of [4, Corollary 4.4], one has to use [4, Theorems 7.1 and 13.1].

If $L_1, ..., L_r$ are of the same form as in Corollary 4.2, then $\sigma(L_i) \cap \sigma(L_j) = \varnothing$ for $i \neq j$ implies the existence of a common monic left multiple of $L_1, ..., L_r$. This result leads to the following conjecture: if $L_1, ..., L_r$ are monic operator polynomials with mutually disjoint spectra, then $L_1, ..., L_r$ have a common monic multiple (cf. Theorem 5.2 and the remark after Corollary 5.3).

5. Common Multiples of Minimal Degree

Let $L_1, ..., L_r$ be monic operator polynomials. Define $\operatorname{ind}_2(L_1, ..., L_r)$ to be the least positive integer k such that $V_m(L_1, ..., L_r)$ has a generalized inverse for $m \geqslant k$. If no such number exists, then we set $\operatorname{ind}_2(L_1, ..., L_r) = \infty$. In the finite-dimensional case, $\operatorname{ind}_2(L_1, ..., L_r)$ is always equal to one, but in the infinite-dimensional case $\operatorname{ind}_2(L_1, ..., L_r)$ may be any positive integer or infinite.

Lemma 5.1. *If* $\operatorname{ind}_1(L_1, ..., L_r) \leqslant k$ *and* $V_k(L_1, ..., L_r)$ *has a generalized inverse, then* $\operatorname{ind}_2(L_1, ..., L_r) \leqslant k$.

Proof. Take $m > k$ and write

$$V_m(L_1, ..., L_r) = \begin{bmatrix} V_k(L_1, ..., L_r) \\ B \end{bmatrix}.$$

The condition $\operatorname{ind}_1(L_1, ..., L_r) \leqslant k$ implies that $\operatorname{Ker} V_k(L_1, ..., L_r) \subset \operatorname{Ker} B$. Hence, as $V_k(L_1, ..., L_r)$ has a generalized inverse, we can apply Lemma 1.2(1) to show that the same is true for $V_m(L_1, ..., L_r)$. But then $\operatorname{ind}_2(L_1, ..., L_r) \leqslant k$.

The following theorem is one of the main results of this paper.

THEOREM 5.2. *If* $\text{ind}_1(L_1, ..., L_r)$ *and* $\text{ind}_2(L_1, ..., L_r)$ *are both finite, then* $L_1, ..., L_r$ *have a common monic left multiple and the least degree of such a multiple is equal to* $\max(\text{ind}_1(L_1, ..., L_r), \text{ind}_2(L_1, ..., L_r))$.

On the other hand, if $L_1, ..., L_r$ *have a common monic left multiple of degree* m *and the spectra of* $L_1, ..., L_r$ *are mutually disjoint, then* $V_m(L_1, ..., L_r)$ *is regular.*

Proof. We prove now only the first part of the theorem. The second part will be proven in Section 9 (see Theorem 9.2).

In view of Theorem 2.2 we may suppose without loss of generality that $\text{ind}_1(L_1, ..., L_r) < m = \text{ind}_2(L_1, ..., L_r)$. Suppose that $L_1, ..., L_r$ have a common monic left multiple of degree strictly less than m. Then there exists such a multiple of degree $m - 1$, and so one can find (see Section 2) an operator X such that $XV_{m-1}(L_1, ..., L_r) = S$, where S is the bottom operator row in $V_m(L_1, ..., L_r)$, i.e.,

$$V_m(L_1, ..., L_r) = \begin{bmatrix} V_{m-1}(L_1, ..., L_r) \\ S \end{bmatrix}.$$

But then we may apply Lemma 1.2(2) to show that $V_{m-1}(L_1, ..., L_r)$ has a generalized inverse. However, this contradicts the fact that $m = \text{ind}_2(L_1, ..., L_r)$. So $L_1, ..., L_r$ have no common monic left multiple of degree strictly less than m. By Theorem 2.2, there exists such a multiple of degree m, and hence the first part of the theorem is proved.

COROLLARY 5.3. *Suppose* $\mathfrak{B}$ *is a Hilbert space and assume that the spectra of* $L_1, ..., L_r$ *are mutually disjoint. Then left invertibility of* $V_k(L_1, ..., L_r)$ *is necessary and sufficient for the existence of a common monic left multiple of* $L_1, ..., L_r$ *of degree* k.

Proof. Apply Theorem 5.2 and recall that in the Hilbert space case regularity is the same as left invertibility.

The question remains open whether in the Hilbert space case $V_k(L_1, ..., L_r)$ is always left invertible if $\text{ind}_1(L_1, ..., L_r) \leqslant k$ and $\sigma(L_i) \cap \sigma(L_j) = \varnothing$ for $i \neq j$.

The following example illustrates the result of Theorem 5.2.

EXAMPLE 5.4. In this example, we construct two linear polynomials $L_1(\lambda) = \lambda I - A$ and $L_2(\lambda) = \lambda I - B$ such that $\text{ind}_1(L_1, L_2) = 2$ and $\text{ind}_2(L_1, L_2) = 3$. So by Theorem 5.2 the polynomials L_1 and L_2 have no common monic left multiple of degree 2 (with bounded coefficients), but such a multiple of degree 3 exists. Furthermore, we show that it is possible to construct a common monic left multiple of L_1 and L_2 of degree 2 if one allows the coefficients of the multiple to be unbounded operators.

Let $\mathfrak{B}$ be the Hilbert space $L_2(0, 1) \oplus L_2(0, 1)$, i.e., $\mathfrak{B}$ is the Hilbert space direct sum of two copies of the Hilbert space of all square Lebesque-integrable functions on $(0, 1)$. Let A and B be the operators on $\mathfrak{B}$ defined by the following matrices:

$$A = \begin{bmatrix} 0 & \alpha t + 1 \\ \beta & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \beta t + 1 \\ \alpha & 0 \end{bmatrix}.$$

Here α and β are different complex numbers and the entries have to be considered as multiplication operators on $L_2(0, 1)$. So for $(\varphi_1, \varphi_2) \in \mathfrak{B} = L_2(0, 1) \oplus L_2(0, 1)$ we have

$$A(\varphi_1, \varphi_2)(t) = ((\alpha t + 1)\, \varphi_2(t), \beta\varphi_1(t)), \qquad 0 < t < 1.$$

A similar formula holds for B.

Note that $\operatorname{Ker}(B - A) = (0)$ and $\operatorname{Im}(B - A)$ is not closed. Further,

$$B^2 - A^2 = \begin{bmatrix} \alpha - \beta & 0 \\ 0 & \alpha - \beta \end{bmatrix}.$$

Therefore, as $\alpha \neq \beta$, the operator $B^2 - A^2$ is invertible.

Put $L_1(\lambda) = \lambda I - A$ and $L_2(\lambda) = \lambda I - B$. As $\operatorname{Ker}(B - A) = (0)$, we have

$$\operatorname{Ker} V_1(L_1, L_2) = \operatorname{Ker} \begin{bmatrix} I & I \\ A & B \end{bmatrix} = (0),$$

and hence $\operatorname{ind}_2(L_1, L_2) = 2$. Note that

$$\begin{bmatrix} I & I \\ A & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B - A \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$

So $\operatorname{Im} V_2(L_1, L_2)$ is not closed. In particular, we see that $V_2(L_1, L_2)$ has no generalized inverse. Next, we observe that

$$\begin{bmatrix} I + (B^2 - A^2)^{-1} A^2 & 0 & -(B^2 - A^2)^{-1} \\ -(B^2 - A^2)^{-1} A^2 & 0 & (B^2 - A^2)^{-1} \end{bmatrix}$$

is a left inverse of $V_3(L_1, L_2)$. Therefore (using Lemma 5.1), we have $\operatorname{ind}_2(L_1, L_2) = 3$.

Hence, by Theorem 5.2, the polynomials L_1 and L_2 have no common monic left multiple of degree 2. In other words, there is no monic operator polynomial of degree 2 which has A and B as right operator roots. However, if one allows unbounded coefficients, then such a polynomial does exist. In fact, A and B

are right roots of the following polynomial of degree 2 with unbounded coefficients:

$$\lambda^2 I + \lambda \begin{bmatrix} 0 & -1 \\ t^{-1} & 0 \end{bmatrix} + \begin{bmatrix} -\alpha\beta t & 0 \\ 0 & -\alpha\beta t - \alpha - \beta - t^{-1} \end{bmatrix}.$$

As $\max(\operatorname{ind}_1(L_1, L_2), \operatorname{ind}_2(L_1, L_2)) = 3$, the polynomials L_1 and L_2 have a common monic left multiple (with bounded coefficients) of degree 3. Using the method described in Section 2, one finds that

$$\lambda^3 I + \lambda^2 \begin{bmatrix} 0 & \alpha\beta t^2 - 1 \\ -\alpha\beta t - \alpha - \beta & 0 \end{bmatrix} + \alpha\beta(\alpha t + 1)(\beta t + 1) \begin{bmatrix} 0 & -t \\ 1 & 0 \end{bmatrix}$$

is such a multiple.

6. Finite Stabilization Index and No Common Multiple

In this section we present two examples. In the first example we construct for each positive integer $n \geqslant 2$ two operators A and B such that

$$\operatorname{ind}_1(\lambda I - A, \lambda I - B) = 2, \qquad \operatorname{ind}_2(\lambda I - A, \lambda I - B) = n + 1.$$

In the second example, we construct two operators Z_1 and Z_2 such that $\operatorname{ind}_1(\lambda I - Z_1, \lambda I - Z_2) = 2$, but Z_1 and Z_2 are not right roots of any monic operator polynomial of any degree.

Example 6.1. Take $n \geqslant 2$, and let $\mathfrak{B}_n$ be the Hilbert space direct sum of n copies of $L_2(0, 1)$. Let α and β be two complex numbers such that

$$\alpha(\beta t + 1)^{n-1} - \beta(\alpha t + 1)^{n-1} \neq 0, \qquad 0 \leqslant t \leqslant 1. \tag{6.1}$$

Let A and B be the operators on $\mathfrak{B}_n$ defined by the following matrices:

$$A = \begin{bmatrix} 0 & (\beta t + 1) I_{n-1} \\ \alpha & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & (\alpha t + 1) I_{n-1} \\ \beta & 0 \end{bmatrix}.$$

Here I_{n-1} denotes the $(n-1) \times (n-1)$ diagonal matrix whose diagonal elements are all equal to one and the entries in the above matrices have to be considered as multiplication operators on $L_2(0, 1)$. In other words, for $0 < t < 1$ we have

$$A(\varphi_1, \varphi_2, \ldots, \varphi_n)(t) = ((\beta t + 1)\,\varphi_2(t), \ldots, (\beta t + 1)\,\varphi_n(t), \alpha\varphi_1(t)),$$
$$B(\varphi_1, \varphi_2, \ldots, \varphi_n)(t) = ((\alpha t + 1)\,\varphi_2(t), \ldots, (\alpha t + 1)\,\varphi_n(t), \beta\varphi_1(t)).$$

Note that for $n = 2$, the operators A and B are equal to the operators introduced in Example 5.4.

Put $L_1(\lambda) = \lambda I - A$ and $L_2(\lambda) = \lambda I - B$. Then

$$V_2(L_1, L_2) = \begin{bmatrix} I & I \\ A & B \end{bmatrix}.$$

As condition (6.1) implies that $\alpha \neq \beta$, one easily sees that $\operatorname{Ker}(B - A) = (0)$. But then $\operatorname{Ker} V_2(L_1, L_2) = (0)$, and thus $\operatorname{ind}_1(L_1, L_2) = 2$.

Next, we compute $\operatorname{ind}_2(L_1, L_2)$. Observe that for $m \geqslant 1$

$$V_{m+1}(L_1, L_2) = \begin{bmatrix} I & I \\ A & B \\ \cdots & \cdots \\ A^m & B^m \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A^m & 0 & \cdots & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B - A \\ \cdots & \cdots \\ 0 & B^m - A^m \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}.$$

From this it follows that $V_{m+1}(L_1, L_2)$ has a left inverse if and only if

$$Z_m = \operatorname{col}(B^k - A^k)_{k=1}^m$$

has a left inverse. Now, for $1 \leqslant k \leqslant n$, we have

$$B^k - A^k = \begin{bmatrix} 0 & \{(\beta t + 1)^k - (\alpha t + 1)^k\} I_{n-k} \\ \{\alpha(\beta t + 1)^{k-1} - \beta(\alpha t + 1)^{k-1}\} I_k & 0 \end{bmatrix}.$$

In particular, one sees that $B^n - A^n = \{\alpha(\beta t + 1)^{n-1} - \beta(\alpha t + 1)^{n-1}\} I_n$. So, by condition (6.1), the operator $B^n - A^n$ is invertible. But then Z_n has a left inverse, and thus the same is true for $V_{n+1}(L_1, L_2)$. As $\operatorname{ind}_1(L_1, L_2) = 2$, this implies that $\operatorname{ind}_2(L_1, L_2) \leqslant n + 1$ (cf. Lemma 5.1).

To prove that $\operatorname{ind}_2(L_1, L_2) = n + 1$, it suffices to show that Z_{n-1} is not left invertible. Observe that for $1 \leqslant k \leqslant n - 1$ and $\varphi \in L_2(0, 1)$

$$(B^k - A^k)(0, \ldots, 0, \varphi)(t) = (0, \ldots, 0, \{(\beta t + 1)^k - (\alpha t + 1)^k\} \varphi(t), 0, \ldots, 0),$$

where on the right-hand side the nonzero entry appears in the $(n - k)$th place. As $(\beta t + 1)^k - (\alpha t + 1)^k$ is equal to zero for $t = 0$, there exists a sequence (φ_n) in $L_2(0, 1)$ such that $\| \varphi_n \| = 1$ for $n = 1, 2, \ldots$ and

$$Z_{n-1}(0, \ldots, 0, \varphi_n) \to 0.$$

This implies that Z_{n-1} is not left invertible. So $\operatorname{ind}_2(L_1, L_2) = n + 1$.

In view of Theorem 5.2, the operators A and B are right operator roots of some monic operator polynomial of degree $n + 1$, but there is no monic operator polynomial of degree strictly less than $n + 1$ which has A and B as right operator roots.

EXAMPLE 6.2. For each $n \geqslant 2$, let $\mathfrak{B}_n$ be the Hilbert space introduced in the previous example, and put $\mathfrak{B} = \mathfrak{B}_2 \oplus \mathfrak{B}_3 \oplus \cdots$, where the direct sum has to be understood as a Hilbert space direct sum. On $\mathfrak{B}$ we define two operators

$$Z_1 = A_2 \oplus A_3 \oplus \cdots, \qquad Z_2 = B_2 \oplus B_3 \oplus \cdots,$$

where for each $n \geqslant 2$ the operators A_n and B_n act on $\mathfrak{B}_n$ and their action is given by

$$A_n = \begin{bmatrix} 0 & (\alpha_n t + 1) I_{n-1} \\ \beta_n & 0 \end{bmatrix}, \qquad B_n = \begin{bmatrix} 0 & (\beta_n t + 1) I_{n-1} \\ \alpha_n & 0 \end{bmatrix}.$$

The complex numbers α_n and β_n are chosen in such a way that for each $n \geqslant 2$

$$\alpha_n(\beta_n t + 1)^{n-1} - \beta_n(\alpha_n t + 1)^{n-1} \neq 0, \qquad 0 \leqslant t \leqslant 1,$$

and the sequences (α_j) and (β_j) are bounded. Because of the last condition Z_1 and Z_2 are well-defined bounded linear operators on $\mathfrak{B}$. We shall prove that Z_1 and Z_2 are not right roots of any monic operator polynomial.

Indeed, suppose the contrary holds true. Then there exists a polynomial $L(\lambda) = \lambda^\ell I + \sum_{j=0}^{\ell-1} \lambda^j C_j$ such that Z_1 and Z_2 are right roots of L. Let P be the orthogonal projection of $\mathfrak{B}$ onto $\mathfrak{B}_\ell$. Then $PZ_i = Z_i P$ and thus

$$0 = PL(Z_i)\, P = (PZ_iP)^\ell + \sum_{j=0}^{\ell-1} (PC_jP)(PZ_iP)^j$$

for $i = 1, 2$. Note that the restriction of PZ_1P to $\mathfrak{B}_\ell$ is equal to A_ℓ and the restriction of PZ_2P to $\mathfrak{B}_\ell$ is B_ℓ. Hence, if for $1 \leqslant j \leqslant \ell - 1$ the operator D_j is defined to be the restriction of PC_jP to $\mathfrak{B}_\ell$, then A_ℓ and B_ℓ are both right roots of the polynomial $\lambda^\ell I + \sum_{j=0}^{\ell-1} \lambda^j D_j$. However, we know from the previous example that this is impossible. So Z_1 and Z_2 are not roots of any monic operator polynomial.

Still, we can prove that $\operatorname{ind}_1(\lambda I - Z_1, \lambda I - Z_2) = 2$ in this case. Indeed, we have only to show that $\operatorname{Ker}(Z_1 - Z_2) = (0)$. Take $x = (x_2, x_3, \ldots) \in \mathfrak{B} = \mathfrak{B}_2 \oplus \mathfrak{B}_3 \oplus \cdots$, and suppose that $(Z_1 - Z_2)x = 0$. Then for each $j \geqslant 2$ we have $(A_j - B_j)\, x_j = 0$. But $\operatorname{Ker}(A_j - B_j) = (0)$ (cf. the previous example), thus $x_j = 0$, and so $x = 0$.

Although there does not exist a monic operator polynomial (with bounded coefficients) which has Z_1 and Z_2 as right roots, the operators Z_1 and Z_2 are right roots of a monic operator polynomial of degree 2 with unbounded coefficients. Indeed, for each $n \geqslant 2$ put

$$C_{1n} = \begin{bmatrix} 0 & -\{(\alpha + \beta)t + 2\} I_{n-2} & 0 \\ 0 & 0 & -1 \\ t^{-1} & 0 & 0 \end{bmatrix}$$

and $C_{2n} = -C_{1n}B_n - B_n^2$. Then $L(\lambda) = \lambda^2 I + \lambda C_1 + C_2$, where

$$C_1 = C_{12} \oplus C_{13} \oplus \cdots, \qquad C_2 = C_{22} \oplus C_{23} \oplus \cdots,$$

has Z_1 and Z_2 as right operator roots.

7. Extension of Admissible Pairs

In the previous sections it was shown how under certain conditions for a given family of monic operator polynomials $L_1, \ldots, L_r$ a common monic left multiple L of minimal degree may be constructed. In this section we present an alternative, more geometric construction which produces in an explicit way a standard pair for such a multiple in terms of standard pairs of the original polynomials $L_1, \ldots, L_r$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach spaces. If $X: \mathfrak{A} \to \mathfrak{B}$ and $T: \mathfrak{A} \to \mathfrak{A}$ are bounded linear operators, then the pair (X, T) will be called an *admissible* pair with *base space* $\mathfrak{A}$. Each standard pair is admissible and the base space of a standard pair is $\mathfrak{B}^\ell$, where ℓ is the degree of the pair. The admissible pair (X, T) is said to be an *extension* of the admissible pair (X_1, T_1) if there exists a regular operator S from the base space of (X_1, T_1) into the base space of (X, T) such that

$$X_1 = XS, \qquad TS = ST_1.$$

Note that the last equality implies that $\operatorname{Im} S$ is invariant under T. A standard pair (X, T) is an extension of the standard pair (X_1, T_1) if and only if the polynomial represented by (X_1, T_1) is a right divisor of the polynomial represented by (X, T). For the finite-dimensional case this is proved in [4, Section 11], but the same proof holds in the infinite-dimensional case (cf. [8] or [1]).

THEOREM 7.1. *Let (X, T) be an admissible pair with base space $\mathfrak{A}$, and suppose that $A = \operatorname{col}(XT^{i-1})_{i=1}^s$ has a generalized inverse. Let P be a projection of $\mathfrak{A}$ along* $\operatorname{Ker} A$. *Then the admissible pair $(X|_{\operatorname{Im} P}, PT|_{\operatorname{Im} P})$ can be extended to a standard pair of degree s.*

Proof. Choose a generalized inverse $A^+: \mathfrak{B}^s \to \mathfrak{A}$ of A, and write $A^+ = [V_1 \cdots V_s]$. Put

$$C = \begin{bmatrix} 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & & I \\ XT^sV_1 & XT^sV_2 & \cdots & XT\,V_s \end{bmatrix}.$$

As $AA^+A = A$, we have $XT^{i-1}A^+A = XT^{i-1}$ for $i = 1, 2, \ldots, s$. So $CA =$

ATA^+A. Now suppose that A^+ has been chosen such that $A^+A = P$. Then $A^+AP = P$ and $AP = A$. So $CA = APTP$, and hence

$$C(A|_{\operatorname{Im} P}) = (A|_{\operatorname{Im} P})(PT|_{\operatorname{Im} P}).$$

Note that $S = A|_{\operatorname{Im} P}$ is a regular operator from $\operatorname{Im} P$ into $\mathfrak{B}^s$. From what we proved above, we see that $CS = S(PT|_{\operatorname{Im} P})$. Further

$$[I\ 0\ \cdots\ 0]S = [I\ 0\ \cdots\ 0]A|_{\operatorname{Im} P} = X|_{\operatorname{Im} P}.$$

So the standard pair $([I\ 0\ \cdots\ 0], C)$ is an extension of $(X|_{\operatorname{Im} P}, PT|_{\operatorname{Im} P})$, and the proof is complete.

Let (X, T) be an admissible pair. By definition, $\operatorname{ind}_1(X, T)$ is the smallest positive integer m such that

$$\operatorname{Ker} \operatorname{col}(XT^{i-1})_{i=1}^{m} = \operatorname{Ker} \operatorname{col}(XT^{i-1})_{i=1}^{m+n}, \qquad n \geqslant 1.$$

If no such number exists, then we define $\operatorname{ind}_1(X, T) = \infty$. We call $\operatorname{ind}_1(X, T)$ the *index of stabilization* of the pair (X, T).

For a standard pair, the index of stabilization is equal to the degree of the pair. To see this, let ℓ be the degree of the standard pair (X, T). Put $A = \operatorname{col}(XT^{i-1})_{i=1}^{\ell}$. Then A is invertible, and so $\operatorname{ind}_1(X, T) \leqslant \ell$. Now suppose that $\operatorname{ind}_1(X, T) = k < \ell$. Put $A_1 = \operatorname{col}(XT^{i-1})_{i=1}^{k}$ and $A_2 = \operatorname{col}(XT^{i-1})_{i=k+1}^{\ell}$. Note that $\operatorname{Ker} A_1 = (0)$ and $A = \operatorname{col}(A_i)_{i=1}^{2}$. Let $A^{-1} = (B_1 B_2)$ be the inverse of A. Then $A_1 B_1 = I_0$, where I_0 denotes the identity operator on $\mathfrak{B}^k$. So A_1 is right invertible. As $\operatorname{Ker} A_1 = (0)$, this implies that A_1 is invertible. But it is easily seen that this is impossible whenever $k < \ell$. So $\operatorname{ind}_1(X, T) = \ell$.

For a family of admissible pairs $(X_1, T_1),\dots, (X_r, T_r)$, we define $\operatorname{ind}_1\{(X_1, T_1),\dots, (X_r, T_r)\}$ to be the index of stabilization of the pair (X, T), where $X = \operatorname{row}(X_j)_{j=1}^{r}$ and $T = \operatorname{diag}(T_j)_{j=1}^{r}$. If $(X_1, T_1),\dots, (X_r, T_r)$ are standard pairs and if $L_1,\dots, L_r$ are the polynomials represented by $(X_1, T_1),\dots, (X_r, T_r)$, respectively, then

$$\operatorname{ind}_1\{(X_1, T_1),\dots, (X_r, T_r)\} = \operatorname{ind}_1(L_1,\dots, L_r). \tag{7.1}$$

THEOREM 7.2. *Let $(X_1, T_1),\dots, (X_r, T_r)$ be admissible pairs with base spaces $\mathfrak{A}_1,\dots, \mathfrak{A}_r$, respectively. Put $X = \operatorname{row}(X_j)_{j=1}^{r}$ and $T = \operatorname{diag}(T_j)_{j=1}^{r}$. Let s be a positive integer such that $A_j = \operatorname{col}(X_j T_j^{i-1})_{i=1}^{s}$ is regular for $j = 1,\dots, r$, $A = \operatorname{col}(XT^{i-1})_{i=1}^{s}$ has a generalized inverse and*

$$\operatorname{ind}_1(X, T) = \operatorname{ind}_1\{(X_1, T_1),\dots, (X_r, T_r)\} \leqslant s.$$

Let P be a projection of $\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ along $\operatorname{Ker} A$. *Then for* $1 \leqslant j \leqslant r$ *the admissible pair* $(X|_{\operatorname{Im} P}, PT|_{\operatorname{Im} P})$ *is an extension of* (X_j, T_j).

Proof. As $\operatorname{Ker} P = \operatorname{Ker} A$ and $s \geqslant \operatorname{ind}(X, T)$, it follows that $\operatorname{Ker} P$ is T-invariant and $X|_{\operatorname{Ker} P} = 0$. So $XP = X$ and $PT = PTP$. Take a fixed j, and let τ_j be the natural embedding from $\mathfrak{A}_j$ into $\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$. From the definitions of X and T it follows that $X\tau_j = X_j$ and $T\tau_j = \tau_j T_j$. Define $S_j\colon \mathfrak{A}_j \to \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ by setting $S_j = P\tau_j$. Then

$$PTS_j = PTP\tau_j = PT\tau_j = P\tau_j T_j = S_j T_j .$$

Further $XS_j = XP\tau_j = X\tau_j = X_j$. So it remains to show that S_j is regular. From what has been proven so far, it follows that $A_j = AS_j$. Hence, as A_j is regular, the same must be true for S_j, and the proof is complete.

We shall now show how the two previous theorems may be used to give an alternative proof of the following result (cf. Theorem 2.2): if $L_1, \ldots, L_r$ are monic operator polynomials, and m is a positive integer such that $m \geqslant \operatorname{ind}_1(L_1, \ldots, L_r)$ and $V_m(L_1, \ldots, L_r)$ has a generalized inverse, then $L_1, \ldots, L_r$ have a common monic left multiple of degree m.

Let $(X_1, T_1), \ldots, (X_r, T_r)$ be standard pairs of $L_1, \ldots, L_r$, respectively. Further, let k_j be the degree of L_j $(1 \leqslant j \leqslant r)$. Put $X = \operatorname{row}(X_j)_{j=1}^r$ and $T = \operatorname{diag}(T_j)_{j=1}^r$. Then

$$A = \operatorname{col}(XT^{i-1})_{i=1}^m = V_m(L_1, \ldots, L_r)[\operatorname{diag}(U_j)_{j=1}^r],$$

where $U_j = [\operatorname{col}(X_j T_j^{i-1})_{i=1}^{k_j}]^{-1}$. So A has a generalized inverse and we may apply Theorem 7.1.

Let P be a projection of $\mathfrak{B}^{k_1} \oplus \cdots \oplus \mathfrak{B}^{k_r}$ along $\operatorname{Ker} A$. By Theorem 7.1, there exists a standard pair (Q, C) of degree m such that (Q, C) is an extension of the pair $(X|_{\operatorname{Im} P}, PT|_{\operatorname{Im} P})$.

Next, we show that we may also apply Theorem 7.2. Using formula (7.1), we have

$$\operatorname{ind}_1(X, T) = \operatorname{ind}\{X_1, T_1), \ldots, (X_r, T_r)\} \leqslant m.$$

In particular, $\operatorname{ind}_1(X_j, T_j) \leqslant m$ for each j. As (X_j, T_j) is a standard pair, this implies that $m \geqslant k_j$. Now $\operatorname{col}(X_j T_j^{i-1})_{i=1}^{k_j}$ is invertible. So $\operatorname{col}(X_j T_j^{i-1})_{i=1}^m$ is left invertible, and hence regular. So we may apply Theorem 7.2. It follows that the pair $(X|_{\operatorname{Im} P}, PT|_{\operatorname{Im} P})$ is an extension of (X_j, T_j) for each j. But then the standard pair (Q, C) is an extension of (X_j, T_j) for each j. Let L be the monic operator polynomial with the standard pair (Q, C). Then L is the desired multiple.

8. The Vandermonde Operator and Supporting Subspaces

Let $L, L_1, \dots, L_r$ be monic operator polynomials with degrees ℓ, $k_1, \dots, k_r$, respectively. In this section we assume that L is a common left multiple of $L_1, \dots, L_r$, i.e., $L_1, \dots, L_r$ are right divisors of L. In that case one may associate with each L_j a so-called supporting subspace $\mathcal{M}_j$. Our aim is to describe the invertibility properties of the Vandermonde operator $V_m(L_1, \dots, L_r)$ in terms of the geometric properties of the spaces $\mathcal{M}_1, \dots, \mathcal{M}_r$.

First, let us recall the definition of the supporting subspaces. Let (X, T), $(X_1, T_1), \dots, (X_r, T_r)$ be standard pairs of $L, L_1, \dots, L_r$, respectively. For each j put

$$K_j = [\operatorname{row}(W_j)_{j=1}^{\ell}] \cdot [\operatorname{col}(X_j T_j^{i-1})_{i=1}^{\ell}] \colon \mathfrak{B}^{k_j} \to \mathfrak{B}^{\ell},$$

where

$$\operatorname{row}(W_j)_{j=1}^{\ell} = [\operatorname{col}(XT^{i-1})_{i=1}^{\ell}]^{-1}.$$

Then

(i) K_j is left invertible;

(ii) $\mathcal{M}_j = \operatorname{Im} K_j$ is invariant under T and $TK_j = K_j T_j$;

(iii) $XT^{\alpha} K_j = X_j T_j^{\alpha}$ for $\alpha = 0, 1, 2, \dots$;

(iv) the map $[\operatorname{col}(XT^{i-1})_{i=1}^{k_j}] \mid_{\mathcal{M}_j} \colon \mathcal{M}_j \to \mathfrak{B}^{k_j}$ is invertible and its inverse is given by $K_j U_j$, where

$$U_j = [\operatorname{col}(X_j T_j^{i-1})_{i=1}^{k_j}]^{-1}.$$

For a proof of these statements, we refer to [4]. The subspace $\mathcal{M}_j$ is called the *supporting subspace* of the right divisor L_j corresponding to the standard pair (X, T) of L. Observe that each $\mathcal{M}_j$ is a complemented subspace of $\mathfrak{B}$. The notations introduced above will be used in this section (and also in the remaining other sections) without further explanation.

By property (iii), mentioned above, the Vandermonde operator $V_m(L_1, \dots, L_r)$ admits the following representation:

$$V_m(L_1, \dots, L_r) = [\operatorname{col}(XT^{i-1})_{i=1}^{m}] \cdot [\operatorname{row}(K_j)_{j=1}^{r}] \operatorname{diag}(U_j)_{j=1}^{r}. \tag{8.1}$$

As $\operatorname{col}(XT^{i-1})_{i=1}^{\ell}$ is invertible, the first factor in the right-hand side of (8.1) will be left invertible for $m \geqslant \ell$. Hence from (8.1) one can easily see (cf. the proof of [4, Theorem 4.1]) that for $m \geqslant \ell$

$$Z_m V_m(L_1, \dots, L_r) = V_m(L_1, \dots, L_r) \cdot \operatorname{diag}(C_j)_{j=1}^{r},$$

where $C_1, \dots, C_r$ are the first companion operators of $L_1, \dots, L_r$ respectively, and $Z_m = \operatorname{col}(XT^{i-1})_{i=1}^m Y$, where Y is some left inverse of $\operatorname{col}(XT^{i-1})_{i=1}^m$. For $m = \ell$ the operator Z_m is merely the first companion operator C of L, and thus

$$CV_\ell(L_1, \ldots, L_r) = V_\ell(L_1, \ldots, L_r) \cdot \operatorname{diag}(C_j)_{j=1}^r.$$

The previous formula also appears in [9]. For the special case that $\ell = k_1 + \cdots + k_r$ and $L_1, \ldots, L_r$ have degree 1 it has been given in [11]. The finite dimensional variant has been discussed in [4, Section 4].

We shall now clarify the relation between the properties of the Vandermonde operator $V_m(L_1, \ldots, L_r)$ and those of the supporting subspaces. From the representation (8.1) it follows that

$$\operatorname{Im} V_m(L_1, \ldots, L_r) = [\operatorname{col}(XT^{i-1})_{i=1}^m](\mathcal{M}_1 + \cdots + \mathcal{M}_r). \tag{8.3}$$

Further for $m \geqslant \ell$

$$\dim \operatorname{Ker} V_m(L_1, \ldots, L_r) = \sum_{j=1}^{r-1} \dim\{(\mathcal{M}_1 + \cdots + \mathcal{M}_j) \cap \mathcal{M}_{j+1}\}. \tag{8.4}$$

In particular, for $m \geqslant \ell$ we have $\operatorname{Ker} V_m(L_1, \ldots, L_r) = (0)$ if and only if the subspaces $\mathcal{M}_1, \ldots, \mathcal{M}_r$ are *linearly independent*, i.e., $x_1 + \cdots + x_r = 0$, $x_j \in \mathcal{M}_j$ for $1 \leqslant j \leqslant r$ implies $x_1 = \cdots = x_r = 0$ (cf. [4, Section 5]).

Note that the third factor in the right-hand side of (8.1) is invertible. So (8.3) is an immediate corollary of (8.1) and the fact that $\operatorname{Im} K_j = \mathcal{M}_j$. We have already observed that for $m \geqslant \ell$ the first factor in the right-hand side of (8.1) is left invertible. So

$$\dim \operatorname{Ker} V_m(L_1, \ldots, L_r) = \dim \operatorname{Ker}[\operatorname{row}(K_j)_{j=1}^r], \qquad m \geqslant \ell.$$

As $\operatorname{Im} K_j = \mathcal{M}_j$ and K_j is injective, this yields (8.4) (cf. [4, Theorem 10.3]).

The sum of linearly independent supporting subspaces does not have to be closed and, when it is closed, it may not have a closed complement. To see this, we consider the following example (cf. Example 2.3).

Let A be an operator acting on an infinite-dimensional Banach space $\mathfrak{A}$, and suppose that $\operatorname{Ker} A = (0)$. Put $L_1(\lambda) = \lambda I$ and $L_2(\lambda) = \lambda I - A$. Then L_1 and L_2 are right divisors of the monic operator polynomial $L(\lambda) = \lambda^2 I - \lambda A$.

Let

$$X = [I \ 0], \qquad C = \begin{bmatrix} 0 & I \\ 0 & A \end{bmatrix}.$$

Then (X, C) is a standard pair of L, and the supporting subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of L_1 and L_2 corresponding to the standard pair (X, C) are given by, respectively,

$$\mathcal{M}_1 = \{(x, y) \in \mathfrak{A}^2 \mid y = 0\}, \qquad \mathcal{M}_2 = \{(x, Ax) \in \mathfrak{A}^2 \mid x \in \mathfrak{A}\}.$$

As $\operatorname{Ker} A = (0)$, it is clear that $\mathscr{M}_1 \cap \mathscr{M}_2 = (0)$, and hence $\mathscr{M}_1$ and $\mathscr{M}_2$ are linearly independent. Note that

$$\mathscr{M}_1 + \mathscr{M}_2 = \mathfrak{A} \oplus \operatorname{Im} C.$$

Thus $\mathscr{M}_1 + \mathscr{M}_2$ is closed (has a closed complement) in $\mathfrak{A}^2$ if and only if $\operatorname{Im} C$ is closed (has a closed complement) in $\mathfrak{A}$. Similarly, $\mathscr{M}_1 + \mathscr{M}_2$ is dense in $\mathfrak{A}^2$ if and only if $\operatorname{Im} C$ is dense in $\mathfrak{A}$.

We shall now state and prove the necessary and sufficient conditions for one-sided invertibility and regularity of $V_m(L_1, ..., L_r)$ in terms of the supporting subspaces $\mathscr{M}_1, ..., \mathscr{M}_r$ of the polynomials $L_1, ..., L_r$. We begin with left invertibility for the case $m \geqslant \ell$.

The Vandermonde operator $V_m(L_1, ..., L_r)$ is left invertible for $m \geqslant \ell$ if and only if the supporting subspaces $\mathscr{M}_1, ..., \mathscr{M}_r$ are linearly independent and $\mathscr{M} = \mathscr{M}_1 + \cdots + \mathscr{M}_r$ is a complemented subspace of $\mathfrak{B}^\ell$. Indeed, in view of formulas (8.3) and (8.4) we only have to show that

$$[\operatorname{col}(XT^{i-1})_{i=1}^m](\mathscr{M})$$

is a complemented subspace of $\mathfrak{B}^m$ if and only if $\mathscr{M}$ is a complemented subspace of $\mathfrak{B}^\ell$. Put $\Omega = \operatorname{col}(XT^{i-1})_{i=1}^m$. Now, as $m \geqslant \ell$, the operator Ω is left invertible. In particular, Ω is a topological isomorphism from $\mathfrak{B}^\ell$ onto $\operatorname{Im} \Omega$. This fact implies that $\mathscr{M}$ is a complemented subspace of $\mathfrak{B}^\ell$ if and only if $\Omega(\mathscr{M})$ is a complemented subspace of $\operatorname{Im} \Omega$. As $\operatorname{Im} \Omega$ is complemented in $\mathfrak{B}^m$, this completes the proof.

In the same way, one can prove that for $m \geqslant \ell$ the Vandermonde operator $V_m(L_1, ..., L_r)$ is regular if and only if $\mathscr{M}_1, ..., \mathscr{M}_r$ are linearly independent and $\mathscr{M}_1 + \cdots + \mathscr{M}_r$ is closed in $\mathfrak{B}^\ell$.

For $m \geqslant \ell$, the Vandermonde operator $V_m(L_1, ..., L_r)$ is right invertible if and only if $m = \ell$, $\mathfrak{B}^\ell = \mathscr{M}_1 + \cdots + \mathscr{M}_r$, and $\operatorname{Ker} V_m(L_1, ..., L_r)$ is a complemented subspace. To see that these conditions are sufficient, assume that $m = \ell$ and $\mathfrak{B}^\ell = \mathscr{M}_1 + \cdots + \mathscr{M}_r$. Then $V_m(L_1, ..., L_r)$ is surjective by formula (8.3). But, together with $\operatorname{Ker} V_m(L_1, ..., L_r)$ is complemented, this implies that $V_m(L_1, ..., L_r)$ is right invertible. Conversely, suppose that $V_m(L_1, ..., L_r)$ is right invertible. Then, trivially, $\operatorname{Ker} V_m(L_1, ..., L_r)$ is complemented and, again using formula (8.3), one sees that $\operatorname{col}(XT^{i-1})_{i-1}^m$ is surjective. As $m \geqslant \ell$, we also know that $\operatorname{col}(XT^{i-1})_{i=1}^m$ is left invertible. So $\operatorname{col}(XT^{i-1})_{i=1}^m$ is invertible. But, since $\operatorname{col}(XT^{i-1})_{i=1}^\ell$ is invertible too, this can only happen if $m = \ell$. Further, using the invertibility of $\operatorname{col}(XT^{i-1})_{i=1}^m$ and the fact that $V_m(L_1, ..., L_r)$ is surjective, it follows from formula (8.3) that $\mathscr{M}_1 + \cdots + \mathscr{M}_r$ must be $\mathfrak{B}^\ell$.

As a consequence of the preceding discussion, we obtain the following result: For $m \geqslant \ell$, the Vandermonde operator $V_m(L_1, ..., L_r)$ is two-sided invertible if and only if $m = \ell$, the supporting subspaces $\mathscr{M}_1, ..., \mathscr{M}_r$ are linearly independent and $\mathscr{M}_1 + \cdots + \mathscr{M}_r = \mathfrak{B}^\ell$.

Note that for $m \geqslant \ell$, the conditions for regularity or left invertibility do not depend on m. It follows that for $m \geqslant \ell$ the Vandermonde operator $V_m(L_1,..., L_r)$ is regular or left invertible if and only $V_\ell(L_1,..., L_r)$ is regular or left invertible.

The case $m \leqslant \ell$ can be treated in an analogous way. Also in this case necessary and sufficient conditions may be given in terms of the supporting subspaces $\mathscr{M}_1,..., \mathscr{M}_r$ in order that the Vandermonde operator $V_m(L_1,..., L_r)$ is one-sided or two-sided invertible. For instance, if $m \leqslant \ell$, then the Vandermonde operator $V_m(L_1,..., L_r)$ is invertible if and only if $\mathscr{M}_1,..., \mathscr{M}_r$ are linearly independent and $\mathscr{M}_1 + \cdots + \mathscr{M}_r$ is the supporting subspace of some monic right divisor L_0 of L of degree m. Furthermore, in that case L_0 is a common monic left multiple of $L_1,..., L_r$ (cf. [4, Section 5]).

Formula (8.1) and the other results in this section may be used to prove the infinite-dimensional versions of [4, Theorems 4.3, 6.1, and 6.2].

9. Spectral Conditions and the Vandermonde Operator

Let $L, L_1,..., L_r$ be monic operator polynomials with degrees $\ell, k_1,..., k_r$, respectively, and assume that $L_1,..., L_r$ are right divisors of L. In this section we study the effect of certain spectral properties of the polynomials $L_1,..., L_r$ on the Vandermonde operator $V_m(L_1,..., L_r)$, especially with regard to invertibility and regularity of $V_m(L_1,..., L_r)$.

Recall that the spectrum $\sigma(L)$ of an operator polynomial L consists of all complex numbers λ such that $L(\lambda)$ is not two-sided invertible. If L is monic, then $\sigma(L)$ coincides with the spectrum (in the usual sense) of the first companion operator of L. In fact, $\sigma(L) = \sigma(T)$, where T may be taken from any standard pair (X, T) of L.

Let $L, L_1,..., L_r$ be as in the first paragraph of this section. Fix a standard pair (X, T) of L, and, as before, let $\mathscr{M}_1,..., \mathscr{M}_r$ be the supporting subspaces of $L_1,..., L_r$ (corresponding to the standard pair (X, T)). Then

$$\sigma(L_i) = \sigma(T|_{\mathscr{M}_i}), \qquad i = 1,..., r. \tag{9.1}$$

This equality follows from the fact that the restriction of T to $\mathscr{M}_i$ is similar to the first companion operator of L_i (see [8]).

We start with an auxiliary result, which is of interest on its own.

Lemma 9.1. *Let S be an operator acting on the Banach space $\mathfrak{A}$, and let $\mathscr{N}_1,..., \mathscr{N}_r$ be S-invariant complemented subspaces of $\mathfrak{A}$. Suppose that*

$$\sigma(S|_{\mathscr{N}_i}) \cap \sigma(S|_{\mathscr{N}_j}) = \varnothing, \qquad (i \neq j).$$

Then $\mathscr{N}_1,..., \mathscr{N}_r$ are linearly independent and $\mathscr{N}_1 \dotplus \cdots \dotplus \mathscr{N}_r$ is closed.

Proof. To prove this lemma, we use a standard construction from operator theory (cf. [2]). Let $\ell_\infty(\mathfrak{A})$ be the Banach space of all bounded sequences $(x_n)_{n=1}^\infty$ with elements in $\mathfrak{A}$ endowed with the supremum norm, and let $c_0(\mathfrak{A})$ be the closed subspace of $\ell_\infty(\mathfrak{A})$ consisting of all sequences in $\mathfrak{A}$ which converge to zero. Let $\tilde{\mathfrak{A}}$ be the quotient space $\ell_\infty(\mathfrak{A})/c_0(\mathfrak{A})$ endowed with the usual quotient norm. An element in $\tilde{\mathfrak{A}}$ will be denoted by $[(x_n)_{n=1}^\infty]$. For each operator S on $\mathfrak{A}$, define $\tilde{S}$ to be operator on $\tilde{\mathfrak{A}}$ given by

$$\tilde{S}[(x_n)_{n=1}^\infty] = [(Sx_n)_{n=1}^\infty].$$

Then $\tilde{S}$ is a well-defined operator on $\tilde{\mathfrak{A}}$ and the map $S \to \tilde{S}$ is an algebraic homomorphism which carries $I_{\mathfrak{A}}$ into $I_{\tilde{\mathfrak{A}}}$.

Let $\mathscr{N}$ be an arbitrary complemented subspace of $\mathfrak{A}$, and let P be a projection of $\mathfrak{A}$ onto $\mathscr{N}$. Then the associated operator $\tilde{P}$ is a projection of $\tilde{\mathfrak{A}}$. Define $\tilde{\mathscr{N}}$ to be the image of $\tilde{P}$. Note that the definition of $\tilde{\mathscr{N}}$ does not depend on the special choice of P. Now assume that $\mathscr{N}$ is invariant under S. Then $SP = PSP$, and thus $\tilde{S}\tilde{P} = \tilde{P}\tilde{S}\tilde{P}$. It follows that $\tilde{\mathscr{N}}$ is invariant under $\tilde{S}$. Furthermore, using standard arguments (cf. [2]), one can show that

$$\sigma(S|_{\mathscr{N}}) = \sigma(\tilde{S}|_{\tilde{\mathscr{N}}}). \tag{9.2}$$

We begin now the proof of the lemma. For $r = 1$, there is nothing to prove. Therefore, take $r \geqslant 2$ and suppose that the lemma has been proved for $r - 1$ spaces. Let

$$J\colon \mathscr{N}_1 \oplus \cdots \oplus \mathscr{N}_r \to \mathfrak{A}$$

be defined by $J(x_1, \ldots, x_r) = x_1 + \cdots + x_r$. We have to show that J is regular. Suppose not. Then there exist $(x_{1n}, \ldots, x_{rn}) \in \mathscr{N}_1 \oplus \cdots \oplus \mathscr{N}_r$, $n = 1, 2, \ldots$, such that $\| x_{1n} \| + \cdots + \| x_{rn} \| = 1$ for $n = 1, 2, \ldots$ and

$$x_{1n} + \cdots + x_{rn} \to 0, \qquad (n \to \infty).$$

By passing to the space $\tilde{\mathfrak{A}}$ we see that it suffices to show that the spaces $\tilde{\mathscr{N}}_1, \ldots, \tilde{\mathscr{N}}_r$ are linearly independent.

From formula (9.2) and the conditions of the lemma, it follows that

$$\sigma(\tilde{S}|_{\tilde{\mathscr{N}}_i}) \cap \sigma(\tilde{S}|_{\tilde{\mathscr{N}}_j}) = \varnothing, \qquad (i \neq j).$$

So by our induction hypothesis, the spaces $\tilde{\mathscr{N}}_1, \ldots, \tilde{\mathscr{N}}_{r-1}$ are linearly independent and $\mathscr{M} = \tilde{\mathscr{N}}_1 + \cdots + \tilde{\mathscr{N}}_{r-1}$ is closed in $\tilde{\mathfrak{A}}$. It remains to prove that $\mathscr{M} \cap \tilde{\mathscr{N}}_r = (0)$. Note that $\mathscr{M}$ is invariant under $\tilde{S}$ and

$$\sigma(\tilde{S}|_{\mathscr{M}}) = \bigcup_{j=1}^{r-1} \sigma(\tilde{S}|_{\tilde{\mathscr{N}}_j}).$$

Hence, $\sigma(\tilde{S}|_{\mathscr{M}}) \cap \sigma(\tilde{S}|_{\mathscr{N}_r})$ is empty. Let $\mathscr{M}_0 = \mathscr{M} \cap \tilde{\mathscr{N}}_r$. Then $\mathscr{M}_0$ is an $\tilde{S}$-invariant closed subspace of $\tilde{\mathfrak{A}}$, and hence

$$\partial\sigma(\tilde{S}|_{\mathscr{M}_0}) \subset \sigma(\tilde{S}|_{\mathscr{M}}) \cap \sigma(\tilde{S}|_{\tilde{\mathscr{N}}_r}) = \varnothing.$$

But then $\mathscr{M}_0 = (0)$, and the proof is complete.

We do not know whether the condition in Lemma 9.1 that the spaces $\mathscr{N}_1,..., \mathscr{N}_r$ are complemented may be replaced by the weaker condition that these spaces are closed.

THEOREM 9.2. *Let $L_1,..., L_r$ be right divisors of the monic operator polynomial L, and let $\mathscr{M}_1,..., \mathscr{M}_r$ be the corresponding supporting subspaces. Suppose that the spectra of $L_1,..., L_r$ are mutually disjoint. Then $\mathscr{M}_1,..., \mathscr{M}_r$ are linearly independent and $\mathscr{M}_1 + \cdots + \mathscr{M}_r$ is closed. Moreover, the Vandermonde operator $V_m(L_1,..., L_r)$ is regular for $m \geqslant$ degree (L).*

Proof. The first part of the theorem follows from formula (9.1) and Lemma 9.1, and the second part is an immediate corollary of the first part (cf. the previous section).

THEOREM 9.3. *Let $L_1,..., L_r$ be right divisors of the monic operator polynomial L, and let $\mathscr{M}_1,..., \mathscr{M}_r$ be the corresponding supporting subspaces. Suppose that the spectra of $L_1,..., L_r$ are mutually disjoint, and let $\bigcup_{j=1}^{r} \sigma(L_j)$ be a relatively open subset of $\sigma(L)$. Then $\mathscr{M}_1,..., \mathscr{M}_r$ are linearly independent and $\mathscr{M}_1 + \cdots + \mathscr{M}_r$ is a complemented subspace. Furthermore, the Vandermonde operator $V_m(L_1,..., L_r)$ is left invertible for $m \geqslant$ degree (L).*

Proof. We already know (see the preceding theorem) that $\mathscr{M}_1,..., \mathscr{M}_r$ are linearly independent and $\mathscr{M} = \mathscr{M}_1 + \cdots + \mathscr{M}_r$ is closed. To prove that $\mathscr{M}$ is complemented, note that by our hypotheses $\sigma(L_1),..., \sigma(L_r)$ are open and closed subsets of $\sigma(L) = \sigma(T)$, where T is taken from the standard pair (X, T) of L. Let $Q_1,..., Q_r$ be the corresponding Riesz projections, i.e., for each j

$$Q_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\lambda I - T)^{-1}\, d\lambda,$$

where the contour Γ_j consists of regular points of T and separates $\sigma(L_j)$ from the rest of the spectrum of $\sigma(L)$. Note that $\mathscr{M}_j \subset \operatorname{Im} Q_j$ because $\operatorname{Im} Q_j$ is the maximal T-invariant subspace $\mathscr{N}$ of $\mathfrak{B}^{\ell}$, where $\ell =$ degree (L), such that $\sigma(T|_{\mathscr{N}})$ lies inside Γ_j. Since $\mathscr{M}_j$ is a complemented subspace of $\mathfrak{B}^{\ell}$, it follows that $\mathscr{M}_j$ is complemented in $\operatorname{Im} Q_j$. The fact that $\sigma(L_1),..., \sigma(L_r)$ are mutually disjoint implies that $\operatorname{Im} Q_1,..., \operatorname{Im} Q_r$ are linearly independent and $\operatorname{Im} Q_1 + \cdots + \operatorname{Im} Q$ is a complemented subspace of $\mathfrak{B}^{\ell}$. As $\mathscr{M} = \mathscr{M}_1 + \cdots + \mathscr{M}_r$ is complemented in $\operatorname{Im} Q_1 + \cdots + \operatorname{Im} Q_r$, it follows that $\mathscr{M}$ is complemented in $\mathfrak{B}^{\ell}$, and the first part of the theorem is proved.

The second part of the theorem is an immediate consequence of the first part (see the previous section).

For the case that $L_1, \dots, L_r$ have degree 1 the parts of Theorems 9.2 and 9.3 dealing with the Vandermonde operator have also been proven by Markus and Mereutsa in [11]. In [11] the standard construction from operator theory used in Lemma 9.1 is employed too. The case of nonlinear divisors also appears in the announcement in [9].

The question remains open whether in Theorem 9.3 the relative openness of $\bigcup_{j=1}^{r} \sigma(L_j)$ in $\sigma(L)$ is a superfluous condition. The answer will be positive if under the conditions of Lemma 9.1 the space $\mathcal{N}_1 + \cdots + \mathcal{N}_r$ is complemented.

10. Spectral Divisors and the Vandermonde Operator

Throughout this section L will be a monic operator polynomial of degree ℓ and $L_1, \dots, L_r$ will be monic right divisors of L. A monic right divisor R of L will be called a (right) *spectral divisor* of L if the spectrum of the quotient corresponding to R is disjoint with the spectrum of R, i.e.,

$$L = QR, \qquad \sigma(Q) \cap \sigma(R) = \varnothing.$$

In that case $\sigma(L)$ is the disjoint union of $\sigma(Q)$ and $\sigma(R)$.

In this section we study the relation between certain spectral conditions on $L_1, \dots, L_r$ and the properties of the Vandermonde operator $V_\ell(L_1, \dots, L_r)$ for the case when $L_1, \dots, L_r$ are spectral divisors.

For divisors of degree 1 the notion of a spectral divisor coincides with the notion of a regular root of the operator equation $L(Z) = 0$, as introduced by Markus and Mereutsa in [11]. Most of the results of this section have been stated and proved in [11] for divisors of degree 1.

As in the finite-dimensional case (see [4, Lemma 7.2]), one can prove that the invertibility of $V_\ell(L_1, \dots, L_r)$ implies that

$$\sigma(LL_j^{-1}) = \bigcup_{i \neq j} \sigma(L_i), \qquad j = 1, \dots, r.$$

This fact leads to the following condition in order that the right divisors $L_1, \dots, L_r$ are spectral: if $V_\ell(L_1, \dots, L_r)$ is invertible and the spectra of $L_1, \dots, L_r$ are mutually disjoint, then $L_1, \dots, L_r$ are spectral divisors of L.

Let (X, T) be a standard pair of L, and let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be the supporting subspaces of $L_1, \dots, L_r$ (corresponding to the standard pair (X, T)). If $L_1, \dots, L_r$ are spectral divisors of L, then (see [7], Theorem 20, and [8])

$$\mathcal{M}_j = \operatorname{Im}\left[\frac{1}{2\pi i} \int_{\Gamma_j} (\lambda I - T)^{-1}\, d\lambda\right], \tag{10.1}$$

where Γ_j is a suitable contour around $\sigma(L_j)$ which separates $\sigma(L_j)$ from $\sigma(L)\backslash\sigma(L_j)$. Using this fact, we shall derive new necessary and sufficient conditions in order that the Vandermonde operator $V_\ell(L_1, ..., L_r)$ is one-sided or two-sided invertible for the case when all divisors $L_1, ..., L_r$ are spectral. As before, these conditions will be phrased in terms of spectra and supporting subspaces.

Let $L_1, ..., L_r$ be right spectral divisors of L, and let $\mathscr{M}_1, ..., \mathscr{M}_r$ be the corresponding supporting subspaces. Then

(α) $\mathscr{M} = \mathscr{M}_1 + \cdots + \mathscr{M}_r$ is a complemented subspace of $\mathfrak{B}^\ell$,

(β) if $\mathscr{M}_i \cap \mathscr{M}_j = (0)$ for $i \neq j$, then $\mathscr{M}_1, ..., \mathscr{M}_r$ are linearly independent.

To see this note that $\sigma(L_1), ..., \sigma(L_r)$ are (relatively) open and closed subsets of $\sigma(T) = \sigma(L)$. Hence the same is true for $\bigcup_{j=1}^r \sigma(L_j)$. Let P be the corresponding Riesz projector, i.e.,

$$P = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} \, d\lambda,$$

where Γ is a suitable contour around $\bigcup_{j=1}^r \sigma(L_j)$, which separates $\bigcup_{j=1}^r \sigma(L_j)$ from $\sigma(L)\backslash\{\bigcup_{j=1}^r \sigma(L_j)\}$. By formula (10.1)

$$\operatorname{Im} P = \mathscr{M}_1 + \cdots + \mathscr{M}_r = \mathscr{M},$$

and (α) is proved. If $\mathscr{M}_i \cap \mathscr{M}_j = (0)$, then $\sigma(L_i) \cap \sigma(L_j) = \varnothing$, because $\mathscr{M}_i \cap \mathscr{M}_j$ is the image of the Riesz projector corresponding to $\sigma(L_i) \cap \sigma(L_j)$. So $\mathscr{M}_i \cap \mathscr{M}_j = (0)$ for $i \neq j$ implies that the spectra of $L_1, ..., L_r$ are mutually disjoint, but then we know from Theorem 9.2 that $\mathscr{M}_1, ..., \mathscr{M}_r$ are linearly independent.

THEOREM 10.1. *Let* $L_1, ..., L_r$ *be right spectral divisors of* L. *Then* Ker $V_\ell(L_1, ..., L_r)$ *is a complemented subspace.*

Proof. Let (X, T) be a standard pair of L, and let $\mathscr{M}_1, ..., \mathscr{M}_r$ be the supporting subspaces of $L_1, ..., L_r$ corresponding to the standard pair (X, T). As $L_1, ..., L_r$ are spectral divisors, the sets $\sigma(L_1), ..., \sigma(L_r)$ are relatively open and closed in $\sigma(L) = \sigma(T)$. For each j, put $\sigma_j = \sigma(L_j)\backslash\{\sigma(L_1) \cup \cdots \cup \sigma(L_{j-1})\}$. Then $\sigma_1, ..., \sigma_r$ are relatively open and closed subsets of $\sigma(T)$ too. Let $\mathscr{N}_j$ be the image of the corresponding Riesz projection, i.e.,

$$\mathscr{N}_j = \operatorname{Im}\left[\frac{1}{2\pi i} \int_{\gamma_j} (\lambda I - T)^{-1} \, d\lambda\right],$$

where γ_j is a suitable contour around σ_j which separates σ_j from $\sigma(L)\backslash\sigma_j$. From formula (10.1) and the functional calculus it follows that

$$\mathscr{M}_j = [\mathscr{M}_j \cap \{\mathscr{M}_1 + \cdots + \mathscr{M}_{j-1}\}] \dotplus \mathscr{N}_j, \qquad 1 \leqslant j \leqslant r. \tag{10.2}$$

Now let us use the notations introduced in Section 8. As $\operatorname{col}(XT^{i-1})_{i=1}^{\ell}$ is invertible, we see from formula (8.1) that

$$\operatorname{Ker} V_\ell(L_1,\dots,L_r) = \operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r].$$

So the proof will be finished if we have shown that the kernel of $\operatorname{row}(K_jU_j)_{j=1}^r$ is complemented. In fact, we shall prove that

$$\operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r] \dotplus \mathscr{D} = \mathfrak{B}^{k_1} \oplus \cdots \oplus \mathfrak{B}^{k_r},$$

where $k_1,\dots,k_r$ are the degrees of $L_1,\dots,L_r$, respectively, and

$$\mathscr{D} = (K_1U_1)^{-1}\mathscr{N}_1 \oplus \cdots \oplus (K_rU_r)^{-1}\mathscr{N}_r.$$

For $1 \leqslant j \leqslant r$ let $x_j \in \mathfrak{B}^{k_j}$, and assume that $x = (x_1,\dots,x_r)$ is a nonzero element of $\operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r]$. Let q be the largest positive integer such that $x_q \neq 0$. Then $\sum_{j=1}^q K_jU_jx_j = 0$, thus, as $\operatorname{Im} K_jU_j = \mathscr{M}_j$, we have

$$K_qU_qx_q \in \mathscr{M}_q \cap \{\mathscr{M}_1 + \cdots + \mathscr{M}_{q-1}\}.$$

Since $\operatorname{Ker} K_qU_q = (0)$ and $x_q \neq 0$, it follows that $K_qU_qx_q$ does not belong to $\mathscr{N}_q$. But then $x \notin \mathscr{D}$. This shows that $\mathscr{D} \cap \operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r] = (0)$.

Next, take $x = (x_1,\dots,x_r) \in \mathfrak{B}^{k_1} \oplus \cdots \oplus \mathfrak{B}^{k_r}$ and assume $x \neq 0$. We want to show that x may be represented as a sum of two vectors, one from $\operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r]$ and one from $\mathscr{D}$. As before, let q be the largest positive integer such that $x_q \neq 0$. We proceed by induction on q.

For $q = 1$, the vector $x \in \mathscr{D}$, because $\mathscr{N}_1 = \mathscr{M}_1$. Take $q \geqslant 2$ and assume that the desired result is correct for smaller integers. As $\operatorname{Im} K_jU_j = \mathscr{M}_j$ for $j = 1,\dots,r$, we see from formula (10.2) that $K_qU_qx_q$ may be written as

$$K_qU_qx_q = K_qU_qz + K_1U_1y_1 + \cdots + K_{q-1}U_{q-1}y_{q-1},$$

where $z \in (K_qU_q)^{-1}(\mathscr{N}_q)$ and $y_j \in \mathfrak{B}^{k_j}$ for $1 \leqslant j \leqslant q-1$. But then

$$\begin{aligned} x = (x_1,\dots,x_{q-1},x_q,0,\dots,0) &= (0,\dots,0,z,0,\dots,0) \\ &+ (-y_1,\dots,-y_{q-1},x_q - z,0,\dots,0) \\ &+ (x_1 + y_1,\dots,x_{q-1} + y_{q-1},0,0,\dots,0). \end{aligned}$$

The first term on the right-hand side is in $\mathscr{D}$, the second term belongs to $\operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r]$ and, by our induction hypothesis, the third term is a sum of a vector in $\mathscr{D}$ and a vector in $\operatorname{Ker}[\operatorname{row}(K_jU_j)_{j=1}^r]$. It follows that x has the desired form, and the proof is complete.

THEOREM 10.2. *Let $L_1, ..., L_r$ be right spectral divisors of L, and let $\mathcal{M}_1, ..., \mathcal{M}_r$ be the corresponding supporting subspaces. Then the following statements are equivalent:*

(i) $V_\ell(L_1, ..., L_r)$ *is left invertible;*

(ii) $\operatorname{Ker} V_\ell(L_1, ..., L_r) = (0)$;

(iii) $\sigma(L_i) \cap \sigma(L_j) = \varnothing$ *for* $i \neq j$;

(iv) $\mathcal{M}_i \cap \mathcal{M}_j = (0)$ *for* $i \neq j$.

Proof. As $L_1, ..., L_r$ are spectral divisors, we know that $\mathcal{M}_1 + \cdots + \mathcal{M}_r$ is a complemented subspace (see the statement (α) in the first part of this section). So (using the results of Section 8) the Vandermonde operator $V_\ell(L_1, ..., L_r)$ is left invertible if and only if $\mathcal{M}_1, ..., \mathcal{M}_r$ are linearly independent. Now, $\mathcal{M}_i \cap \mathcal{M}_j = (0)$ for $i \neq j$ implies that $\mathcal{M}_1, ..., \mathcal{M}_r$ are linearly independent (see statement (β) in the first part of this section). So (iv) $\Rightarrow$ (i).

Clearly, (i) $\Rightarrow$ (ii). Further, by formula (8.4), statement (ii) implies (iv). As $\mathcal{M}_i \cap \mathcal{M}_j$ is the image of the Riesz projector corresponding to $\sigma(L_i) \cap \sigma(L_j)$ we also have (iii) $\Leftrightarrow$ (iv).

THEOREM 10.3. *Let $L_1, ..., L_r$ be right spectral divisors of L, and let $\mathcal{M}_1, ..., \mathcal{M}_r$ be the corresponding supporting subspaces. Then the following statements are equivalent:*

(i) $V_\ell(L_1, ..., L_r)$ *is right invertible;*

(ii) $\operatorname{Ker} V_\ell(L_1, ..., L_r)^* = (0)$;

(iii) $\sigma(L) = \bigcup_{j=1}^r \sigma(L_j)$;

(iv) $\mathcal{M}_1 + \cdots + \mathcal{M}_r = \mathfrak{B}^\ell$.

Proof. Using Theorem 10.1 and the necessary and sufficient condition for right invertibility of the Vandermonde operator, proven in Section 8, one easily sees that (i) $\Leftrightarrow$ (iv). Further, by formula (8.3), statement (ii) is equivalent to $\mathcal{M}_1 + \cdots + \mathcal{M}_r$ is dense in $\mathfrak{B}^\ell$. But, as $L_1, ..., L_r$ are spectral divisors, $\mathcal{M}_1 + \cdots + \mathcal{M}_r$ is closed. So (ii) $\Leftrightarrow$ (iv). Finally, using formula (10.1), we have (iii) $\Leftrightarrow$ (iv).

Combining Theorems 10.2 and 10.3 we obtain the following characterizations of two-sided invertibility of the Vandermonde operator: if $L_1, ..., L_r$ are right spectral divisors of L, and if $\mathcal{M}_1, ..., \mathcal{M}_r$ are the corresponding supporting subspaces, then each of the following statements is equivalent to the two-sided invertibility of the Vandermonde operator $V_\ell(L_1, ..., L_r)$:

(1) $\operatorname{Ker} V_\ell(L_1, ..., L_r) = (0)$ and $\operatorname{Ker} V_\ell(L_1, ..., L_r)^* = (0)$;

(2) $\sigma(L_i) \cap \sigma(L_j) = \varnothing$ for $i \neq j$ and $\sigma(L) = \bigcup_{j=1}^r \sigma(L_j)$;

(3) $\mathcal{M}_i \cap \mathcal{M}_j = (0)$ for $i \neq j$ and $\mathcal{M}_1 + \cdots + \mathcal{M}_r = \mathfrak{B}^\ell$.

11. Similarity and Decomposition of Operator Polynomials

In this section we extend the main result of Section 8 in [4] to the infinite-dimensional case. Two monic operator polynomials L_1 and L_2 are called *similar* if the (first) companion operators of L_1 and L_2 are similar. This definition is equivalent to the following: L_1 and L_2 are similar if there exist standard triples (X_1, T_1, Y_1) and (X_2, T_2, Y_2) (see [6, 8]) for L_1 and L_2 such that

$$X_2 = X_1 H, \qquad T_1 = T_2, \qquad Y_2 = H^{-1} Y$$

for some invertible operator H. In the finite-dimensional case L_1 and L_2 are similar if and only if L_1 and L_2 are polynomially equivalent.

The proof of the following theorem is (except for some references to [6], which have to be replaced by references to [8]) the same as the proof of [4, Theorem 8.2].

THEOREM 11.1. *Let $L, L_1, \ldots, L_r$ be monic operator polynomials with degrees ℓ, $k_1, \ldots, k_r$, respectively, and suppose that $\ell = k_1 + \cdots + k_r$. Further, suppose that $L_1, \ldots, L_r$ are right divisors of L and that for $2 \leqslant j \leqslant r$ the Vandermonde operator $V_{m_j}(L_1, \ldots, L_j)$, where $m_j = k_1 + \cdots + k_j$, is invertible. Then there exists monic operator polynomials $Q_2, \ldots, Q_r$ such that*

$$L(\lambda) = Q_r(\lambda)\, Q_{r-1}(\lambda) \cdots Q_2(\lambda)\, L_1(\lambda)$$

and for $2 \leqslant j \leqslant r$ the polynomials Q_j and L_j are similar.

For divisors of degree 1 the previous theorem was proved by Mereutsa [12] under somewhat more restrictive conditions.

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